SOME CONJECTURES AND PROBLEMS IN COMPLEX GEOMETRY (Open Problems in Complex Geometry)

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Abstract

I would like present some conjectures and open problems in complex algebraic geometry. This report is based on my talk in RIMS workshop in September 2010.

Contents

1 Introduction

In this paper, I would like to present several conjectures and problems in complex geometry. Some of them are well known, but many of them are original.

I have been working in the classification theory of algebraic varieties. The central problem in the classification theory is the minimal model program. The minimal model program is the program to describe the geometry of projective varieties in terms of the following 3 geometries:

(1) Geometry of $\mathbb{Q}$-Fano varieites,

(2) Geometry of varieties with numerically trivial canonical divisors,

(3) Geometry of KLT pairs of log general type.

Minimal model program is divided into two parts.

1.1 Minimal model conjecture

Let X be a smooth projective variety. To single out the geometry of $\mathbb{Q}$-Fano varieties, we need to solve the following conjecture.

Conjecture 1.1 *(Minimal model conjecture) Let X be a smooth projective variety defined over $\mathbb{C}$. Then one of the followings holds.*

(1) X is uniruled,

(2) X is birational to a X_{min} such that

(a) X_{min} is $\mathbb{Q}$-factorial and has only terminal singularities,

(b) $K_{X_{min}}$ is nef.

□

Recently a lot of progress has been made in the minimal model program. The most outstanding recent progress is the proof of the existence of minimal models for KLT pairs of log general type ([B-C-H-M]).

Theorem 1.2 *Let (X, Δ) be a KLT pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi : X \longrightarrow U$ be a projective morphism of quasiprojective varieties.*

If either Δ is π-big and $K_X + \Delta$ is π-pseudo-effective, then

(1) $K_X + \Delta$ has a log terminal model over U,

(2) If $K_X + \Delta$ is π-big, then $K_X + \Delta$ has a log canonical model over U,

(3) If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the $\mathcal{O}_U$-module

$$\mathcal{R}(\pi, \Delta) = \oplus_{m=0}^{\infty} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated. □

But still we do not know how to construct minimal models in the case of non general type. Moreover the minimal model is not so useful, unless the abundance conjecture below holds.

1.2 Abundance conjecture

Let X be a minimal algebraic variety. To describe the geometry of X in terms of geometry of varieties with numerically trivial canonical divisors and geometry of KLT pairs of log general type, we need to prove the following conjectures.

Conjecture 1.3 *(Nonvanishing conjecture ([M])) Let (X, D) be a KLT pair. Suppose that $K_X + D$ is pseudoeffective. Then there exists a positive integer m_0 such that $|m_0(K_X + D)| \neq \emptyset$.* □

Slightly stronger version is the usual abundance conjecture:

Conjecture 1.4 *(Abundance conjecture ([M])) Let (X, D) be a smooth projective variety such that $K_X + D$ is pseudoeffective. Then*

$$\limsup_{m \longrightarrow \infty} \frac{\log h^0(X, m(K_X + D))}{\log m} = \sup_A \limsup_{m \longrightarrow \infty} \frac{\log h^0(X, A + m(K_X + D))}{\log m}$$

holds, where A runs ample line bundle on X and m runs positive integers such that $m(K_X + D)$ is Cartier. □

The main interest of these conjectures are the fact that the conjectures do not assume the strict positivity of the log canonical bundles. Recently C. Birkar proved that Conjecture 1.3 implies Conjecture 1.1.

1.3 Present situation of the abundance conjecture

As far as I know Conjecutres 1.3 and 1.4 are still wide open. Apparently Conjecture 1.3 is exactly the essential part of Conjecture 1.4.

Recently Y.-T. Siu posted his paper [S3] and claimed the proof of Conjectures 1.3 and 1.4 at least in the case of $D = 0$. But unfortunately I cannot understand his argument and it seems to be quite imcomplete.

2 An approach to the abundance conjecture

In this section, I would like to show some idea to prove Conjecture 1.3.

2.1 Canonical metrics on canonical bundles

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman volume form $K_\Omega(z)$ is characterized as the supremum of the square norm $|\eta(z)|^2$ of the element of $A^2(\Omega, K_\Omega)$ with $\| \eta \| = 1$, where $\| \eta \|$ denotes the L^2-norm with respect to the inner product:

$$(\eta, \eta') := (\sqrt{-1})^{n^2} \int_\Omega \eta \wedge \overline{\eta'}.$$

We note that $\log |\eta(z)|^2$ is plurisubharmonic on Ω, where we have identified $|\eta(z)|^2$ with the function

$$\frac{|\eta(z)|^2}{|dz_1 \wedge \cdots dz_n|^2}.$$

Now we set

(2.1.1)

$$dV_{max}(\Omega)(z) := \sup \left\{ dV \,\middle|\, dV = e^{-\varphi} \cdot |dz_1 \wedge \cdots \wedge dz_n|^2, \varphi \in PSH(\Omega), \int_\Omega dV = 1 \right\},$$

where $PSH(\Omega)$ denotes the set of plurisubharmonic functions on Ω. By the sub-mean-value inequality for plurisubharmonic functions, $dV_{max}(\Omega)$. By definition it is clear that $dV_{max} \geqq K_\Omega$ holds.

Let X be a compact complex manifold and let (L, h_L) be a singular hermitian line bundle with semipositive curvature. We shall assume that (L, h_L) is KLT, i.e., the multiplier ideal sheaf $\mathcal{I}(h_L)$ is trivial on X. We set

(2.1.2) $dV_{max}(L, h_L) :=$ the upper semicontinuous envelope of

$$\sup\left\{h^{-1} \mid h\text{: a sing. herm. metric on } K_X \text{ s.t. } \sqrt{-1}\,(\Theta_h + \Theta_{h_L}) \geqq 0,\ \int_X h^{-1} = 1\right\},$$

where sup means the poitwise supremum. We call $dV_{max}(L, h_L)$ the maximal volume form of X with respect to (L, h_L). Then it is easy to see that $h_{min} := dV_{max}(L, h_L)^{-1} \cdot h_L$ is an AZD of $K_X + L$ with minimal singularities. This definition can be generalized to the case of a (not necessarily compact) complex manifold X and a KLT singular hermitian line bundle (L, h_L) on X with semipositive curvature such that

(2.1.3)
$$\left\{h \mid \text{a sing. herm. metric on } K_X \text{ s.t. } \sqrt{-1}\,(\Theta_h + \Theta_{h_L|X}) \geqq 0,\ \int_X h^{-1} = 1\right\} \neq \emptyset.$$

If L is a trivial line bundle $\mathcal{O}_X$ on a complex manifold X and $h_L = 1$, we shall denote $dV_{max}(\mathcal{O}_X, 1)$ simply by $dV_{max}(X)$ and call it the maximal volume form on X. Then for a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, $dV_{max}(\Omega)$ on Ω is a similar object to the Bergman volume form or the Kähler-Einstein volume form on Ω. It seems to be interesting to study $dV_{max}(\Omega)$ even in this special case.

Let $f : X \longrightarrow S$ be a surjective proper Kähler morphism with connected fibers between connected complex manifolds.

Definition 2.1 *Let (L, h_L) be a singular hermitian $\mathbb{Q}$-line bundle on X with semipositive curvature. (L, h_L) is said to be* **KLT**, *if the multiplier ideal $\mathcal{I}(h_L)$ is the structure sheaf $\mathcal{O}_X$.* □

Suppose that $K_{X_s} + L|X_s$ is pseudoeffective for every $s \in S^\circ$. We set

$$S^\circ := \{s \in S \,|\, f \text{ is smooth over } s \text{ and } (L, h_L)|X_s \text{ is well defined and KLT}\}.$$

Suppse that S° is nonempty. We shall consider an analogy of $\hat{h}_{can}$ as follows. For $s \in S^\circ$ we set

(2.1.4) $dV_{max}((L, h_L)|X_s) :=$ the upper semicontinuous envelope of

$$\sup\left\{h^{-1} \mid h\text{: a sing. herm. metric on } K_{X_s} \text{ s.t. } \sqrt{-1}\,(\Theta_h + \Theta_{h_L|X_s}) \geqq 0,\ \int_{X_s} h^{-1} = 1\right\},$$

where sup means the poitwise supremum. We call $dV_{max}((L, h_L)|X_s)$ the maximal volume form of X_s with respect to $(L, h_L)|X_s$. And we set

(2.1.5)
$$h_{min} := dV_{max}^{-1} \cdot h_L.$$

Then it is easy to see that $h_{min,s} := dV_{max}((L, h_L)|X_s)^{-1} \cdot h_L$ is an AZD of $K_{X_s} + L|X_s$ with minimal singularities (see Definition ?? and Section ??). This definition can be generalized to the case of a (not necessarily compact) complex manifold X and a singular hermitian line bundle (L, h_L) on X with semipositive curvature such that

(2.1.6)
$$\left\{h \mid \text{a sing. herm. metric on } K_X \text{ s.t. } \sqrt{-1}\,(\Theta_h + \Theta_{h_L|X}) \geqq 0,\ \int_X h^{-1} = 1\right\} \neq \emptyset.$$

If L is a trivial line bundle $\mathcal{O}_X$ on a complex manifold X and $h_L = 1$, we shall denote $dV_{max}(\mathcal{O}_X, 1)$ simply by $dV_{max}(X)$ and call it the maximal volume form on X. Then for a bounded pseudoconvex domain Ω in $\mathbb{C}^n$, $dV_{max}(\Omega)$ on Ω is a similar object to the Bergman volume form or the Kähler-Einstein volume form on Ω. It seems to be interesting to study $dV_{max}(\Omega)$ even in this special case.

Let T be a closed semipositive current on a compact complex manifold X. We may also consider

(2.1.7) $\quad dV_{max}(T) :=$ the upper semicontinuous envelope of

$$\sup\left\{h^{-1} \mid h\text{: a sing. herm. metric on } K_X \text{ s.t. } \sqrt{-1}\,(\Theta_h + T) \geqq 0,\ \int_{X_s} h^{-1} = 1\right\}.$$

Let X be a compact Kähler manifold and let T be a closed semipositive current on X. Suppose that $c_1(K_X) + T$ is pseudoeffective. Let ω be a Kähler form on X. Then we see that $\{dV_{max}(T + t\omega)\}_{t>0}$ is monotone increasing with respect to t by definition. We set

(2.1.8)
$$d\hat{V}_{max}(T) := \text{the upper semicontinuous envelope of } \lim_{t\downarrow 0} dV_{max}(T + t\omega).$$

We call $d\hat{V}_{max}$ the super maximal volume form associacted with T. It is easy to see this definition does not depend on the choice of the Kähler form ω. It is clear that $d\hat{V}_{max}(T) \geqq dV_{max}(T)$ holds. For a pseudoeffective singular hermitian $\mathbb{Q}$-line bundle (L, h_L) we define $d\hat{V}_{max}(L, h_L)$ similarly as ([?]). And in this case we set

$$\hat{h}_{min} := d\hat{V}_{max}(L, h_L)^{-1} \cdot h_L. \tag{2.1.9}$$

We call $\hat{h}_{min}$ the superminimal metric on $K_X + L$. If (L, h_L) is KLT, then $\hat{h}_{min}$ is an AZD of $K_X + L$.

Now I would like to propose the following conjecture.

Conjecture 2.2 *In the above notations, we define the relative volume form $dV_{max,X/S}(L, h_L)$ (resp. $d\hat{V}_{max,X/S}(L, h_L)$) on $f^{-1}(S^\circ)$ by $dV_{max,X/S}(L, h_L)|X_s := dV_{max}((L, h_L)|X_s)$ ($d\hat{V}_{max,X/S}(L, h_L)|X_s := d\hat{V}_{max}((L, h_L)|X_s)$) for $s \in S^\circ$. And we define the singular hermitian metric $h_{X/S}$ on $(K_{X/S} + L)|f^{-1}(S^\circ)$ by*

(2.1.10)
$$h_{min,X/S}(L, h_L) := \textit{the lower semicontinuous envelope of } dV_{max,X/S}(L, h_L)^{-1} \cdot h_L.$$

Then $h_{min,X/S}(L, h_L)$ extends to a singular hermitian metric on $K_{X/S} + L$ over X and has semipositive curvature. □

We call $h_{min,X/S}$ the minimal singular hemitian metric on $K_{X/S} + L$ with respect to h_L. This conjecture is very similar to Theorem 2.5 below and the recent result of Berndtsson([Ber]). If this conjecture is affirmative, we can prove the defomation ivariance of plurigenera for Kähler deformations. One can consider also the case that (L, h_L) is pseudoeffective with KLT singularities. The following theorem is a supporting evidence for Conjecture 2.2.

Theorem 2.3 *([T6]) Let $f : X \longrightarrow S$ be a proper projective morphism with connected fibers. Let (L, h_L) be a pseudoeffective singular hermitian line bundle on X such that*

$S^\circ := \{s \in S \,|\, f$ *is smooth over* s *and* $(X_s, (L, h_L)|X_s)$ *is well defined and KLT*$\}$

is nonempty. Then

(1) $h(L, h_L)_{min,X/S}$ extends to a singular hermitian metric on the whole $K_{X/S}+$ L with semipositive curvature.

(2) $h(L, h_L)_{min,X/S}|X_s$ is an AZD with minimal singularities on $K_{X_s} + L|X_s$ for every $s \in S^\circ$.

□

2.2 Supercanonical AZD and maximal volume forms

Let (L, h_L) be a KLT singular hermitian $\mathbb{Q}$-line bundle on a smooth projective variety X. Suppose that $K_X + L$ is pseudoeffective. Let A be a sufficiently ample line bundle on X such that for an arbitrary pseudoeffective singular hermitian line bundle $(F.h_F)$ on X, $\mathcal{O}_X(F) \otimes \mathcal{I}(h_F)$ is globally generated. And let h_A be a C^∞-hermitian metric on A. For a positive integer m such that mL is a genuine line bundle and $\sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + L)))$, we set

$$\text{(2.2.1)} \qquad \parallel \sigma \parallel_{\frac{1}{m}} := \left| \int_X h_A^{\frac{1}{m}} \cdot h_L \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right|^{\frac{m}{2}}.$$

For $x \in X$, we set

(2.2.2)
$$\hat{K}_m^A(x) := \sup \left\{ |\, \sigma \,|^{\frac{2}{m}} (x) \mid \sigma \in \Gamma(X, \mathcal{O}_X(A + m(K_X + L))), \parallel \sigma \parallel_{\frac{1}{m}} = 1 \right\}.$$

We note that $\parallel \sigma \parallel_{\frac{1}{m}}$ is well defined by the assumption that (L, h_L) is KLT. We set

$$\text{(2.2.3)} \qquad \hat{h}_{can,A}(L, h_L) := \text{the lower envelope of } \liminf_{m \longrightarrow \infty} h_A^{-\frac{1}{m}} \cdot (\hat{K}_m^A)^{-1}$$

and

$$\text{(2.2.4)} \qquad \hat{h}_{can}(L, h_L) := \text{the lower envelope of } \inf_A \hat{h}_{can,A}(L, h_L),$$

where A runs all the ample line bundles on X.

Theorem 2.4 *([T4]) $\hat{h}_{can,A}(L, h_L)$ and $\hat{h}_{can}(L, h_L)$ defined respectively as (2.2.3) and (2.2.4) are AZD's of $K_X + L$ with minimal singularities. We call $\hat{h}_{can}(L, h_L)$ the supercanonical AZD on $K_X + L$ with respect to h_L.* □

The following theorem is similar to Conjecture 2.2 in the case of supercanonical AZD $\hat{h}_{can}(L, h_L)$.

Theorem 2.5 *([T1, T4]) Let $f : X \longrightarrow S$ be a proper surjective projective morphism between complex manifolds with connected fibers and let (L, h_L) be a pseudoeffective singular hermitian $\mathbb{Q}$-line bundle on X such that for a general fiber X_s, $(L, h_L)|X_s$ is KLT. We set*

$$S^\circ := \{s \in S |\ f \text{ is smooth over } s \text{ and } (L, h_L)|X_s \text{ is well defined and KLT}\}.$$

Then there exists a singular hermitian metric $\hat{h}_{can}(L, h_L)$ on $K_{X/S} + L$ such that

(1) $\hat{h}_{can}(L, h_L)$ has semipositive curvature current,

(2) $\hat{h}_{can}(L, h_L)\,|X_s$ is an AZD on $K_{X_s} + L_s$ (with minimal singularities) for every $s \in S^\circ$,

(3) For every $s \in S^\circ$, $\hat{h}_{can}(L, h_L) \mid X_s \leqq \hat{h}_{can}((L, h_L)|X_s)$ holds, where $\hat{h}_{can}((L, h_L)|X_s)$ denotes the supercanonical AZD on $K_{X_s} + L_s$ with respect to $h_L|X_s$ (cf. Theorem 2.4). And $\hat{h}_{can}(L, h_L)|X_s = \hat{h}_{can}((L, h_L)|X_s)$ holds outside of a set of measure 0 on X_s for almost every $s \in S^\circ$. □

The following conjecture seems to be reasonable.

Conjecture 2.6 *Let X be a smooth projective variety and let (L, h_L) be a pseudoeffective KLT line bundle on X. Then*

$$\hat{h}_{can}(L, h_L) = \hat{h}_{min}(L, h_L) = h_{min}(L, h_L)$$

hold. □

To see the relation between Conjectures 1.3 and 2.6.

Definition 2.7 *Let X be a smooth projective variety and let (L, h_L) be a pseudoeffective KLT line bundle on X. Let $x \in X$ be an arbitrary point. We say that a singular volume form dV on X is an extremal with respect to (L, h_L), if the followings hold:*

(1) $dV^{-1} \otimes h_L$ is a singular hermitian metric on $K_X + L$ with semipositive curvature,

(2) $dV(x) = dV_{max}(x)$. □

The following conjecture asserts that the extremal volume form is generically pluriharmonic.

Conjecture 2.8 *Let X be a smooth projective variety and let D be an effective $\mathbb{Q}$-divisor such that (X, D) is KLT. Let σ_D be a nontrivial multivalued holomorhic section of D with divisor D. We set $h_D := |\sigma_D|^{-2}$. Suppose that $K_X + D$ is pseudoeffective. Then we have the followings:*

(1) *For every extremal volume form dV with respect to (D, h_D), $dV \otimes h_D^{-1}$ is generically pluriharmonic, i.e., the current $\sqrt{-1}\,\Theta_{dV^{-1}\cdot h_D}$ has the vanishing absolutely continuous part.*

(2) *$K_X + D$ is $\mathbb{Q}$-effective.* □

The assertion (1) of Conjecture 2.8 implies that $c_1(K_X + D)$ is numerically equivalent to an infinite sum of prime divisors with nonnegative coefficient. The assertion (2) of Conjecture 2.8 is nothing but Conjecture 1.3 and it will be obtained by using the argument in [Ka2].

2.3 Canonical measures and the minimal volume forms

Let X be a smooth projective variety with nonnegative Kodaira dimension. In [S-T, T3]. Let $f : X \dashrightarrow Y$ be the Iitaka fibration. By taking a suitable modification of X, we may assume that the followings holds:

(1) f is a morphism.

(2) $(f_*\mathcal{O}_X(K_{X/Y}^{\otimes m!}))^{**}$ is an invertible sheaf for every sufficiently large m.

For such a sufficiently large m,

$$L_{X/Y} = \frac{1}{m!}(f_*\mathcal{O}_X(K_{X/Y}^{\otimes m!}))^{**} \tag{2.3.1}$$

is independent of m and we call it the Hodge $\mathbb{Q}$-line bundle. We set

$$h_{X/Y}(\sigma,\sigma)_s^{m!} := \left(\int_{X_s} |\sigma|^{\frac{2}{m!}}\right)^{m!} \quad (\sigma \in L_{X/Y,s}^{\otimes m!}) \tag{2.3.2}$$

and call it the Hodge metric on $L_{X/Y}$. We consider the following equation:

$$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\Theta_{h_{X/Y}} = \omega_Y. \tag{2.3.3}$$

on Y. We see that there exists a unique closed semipositive current solution ω_Y of (2.3.3) such that

(1) There exists a nonempty Zariski open subset U of Y such that $\omega_Y|U$ is a C^∞-Kähler form,

(2) $(\omega_Y^n)^{-1} \cdot h_{L_{X/Y}}$ is an AZD of $K_Y + L_{X/Y}$.

For the proof see [S-T, T3, T5].

Definition 2.9 *We define the volume form:*

$$d\mu_{can} := f^*\left(\omega_Y^n \cdot h_{X/Y}^{-1}\right)$$

on X and call it the canonical measure on X. □

The following conjecture seems to be reasonable.

Conjecture 2.10 *Let X be a smooth projective variety with pseudoeffective canonical bundle. Then*

(1) X has nonnegative Kodaira dimension,

(2)

$$d\hat{V}_{can} = d\hat{V}_{max} = dV_{max} = C \cdot d\mu_{can}$$

hold, where C is a positive constant. □

In this case, the key point is to show that dV_{max} is a kind of Perron solution for the Monge-Ampère equation associated with (2.3.3).

2.4 Canonical volume forms on open manifolds

The construction in Section 2.2 can be generalized to an arbitrary complex manifold. This is just a formal generalization. But it arises the many interesting problems and also is important to consider the degeneration, This subsection is not essential in the later argument. Hence one may skip it. Let M be a complex manifold. For every positive integer m, we set

$$Z_m := \left\{ \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)); \left| \int_M (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| < +\infty \right\}$$

and

$$K_{M,m} := \sup \left\{ |\sigma|^{\frac{2}{m}} \, ; \, \sigma \in \Gamma(M, \mathcal{O}_M(mK_M)), \left| \int_M (\sigma \wedge \bar{\sigma})^{\frac{1}{m}} \right| \leqq 1 \right\},$$

where sup denotes the pointwise supremum.

Proposition 2.11 *[T4]*

$$K_{M,\infty} := \limsup_{m \longrightarrow \infty} K_{M,m}$$

exists and if $Z_m \neq 0$ for some $m > 0$, then $K_{M,\infty}$ is not identically 0 and

$$h_{can,M} := \textit{the lower envelope of } \frac{1}{K_{M,\infty}}$$

is a well defined singular hermitian metric on K_M with semipositive curvature current. □

By definition, $h_{can,M}$ is invariant under the automorphism group $\mathrm{Aut}(M)$. Hence we obtain the following:

Proposition 2.12 *Let Ω be a homogeneous bounded domain in $\mathbb{C}^n$. Then $h^{-1}_{can,\Omega}$ is a constant multiple of the Bergman volume form on Ω.* □

For a general bouded domain in $\mathbb{C}^n$ it seems to be very difficult to calculate the invariant volume form h^{-1}_{can}. Let us consider the punctured disk

$$\Delta^* := \{t \in \mathbb{C} | \; 0 < |t| < 1\}.$$

Then one sees that unlike the Bergman kernel, h_{can,Δ^*} reflects the puncture. The following conjecture seems to be very plausible. But at this moment I do not know how to solve.

Conjecture 2.13

$$h_{can,\Delta^*}^{-1} = O\left(\frac{\sqrt{-1}dt \wedge d\bar{t}}{|t|^2 (\log|t|)^2}\right)$$

holds. □

Conjecture 2.13 is very important in many senses.

Next we shall consider the following situation. Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. We set $M := X\backslash D$. Let A be a sufficiently ample line bundle on X. Let h_A be a C^∞-hermitian metric on A with strictly positive curvature. We define

$$\hat{K}_m^A := \sup\{|\sigma|^{\frac{2}{m}};\ \sigma \in \Gamma(M, \mathcal{O}_M(A + mK_M)),\ \|\sigma\|_{\frac{1}{m}} = 1\}, \tag{2.4.1}$$

where

$$\|\sigma\|_{\frac{1}{m}} := \left|\int_X h_A^{\frac{1}{m}} \cdot (\sigma \wedge \bar{\sigma})^{\frac{1}{m}}\right|^{\frac{m}{2}}. \tag{2.4.2}$$

And we define

$$\hat{h}_{can,A} := \text{the lower envelope of } \liminf_{m \longrightarrow \infty} (\hat{K}_m^A)^{-1}. \tag{2.4.3}$$

We see that $\hat{h}_{can,A}$ is independent of the choice of h_A. We set

$$\hat{h}_{can,M} := \text{the lower envelope of } \inf_A \hat{h}_{can,A},$$

where A runs all the ample line bundle on X. We note that

$$\{\sigma \in \Gamma(M, \mathcal{O}_M(A + mK_M)),\ \|\sigma\|_{\frac{1}{m}} < \infty\} \simeq \Gamma(X, \mathcal{O}_X(mK_X + (m-1)D))$$

holds by a simple calculation.

Definition 2.14 *Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. Let σ_D be a global holomorphic section of $\mathcal{O}_X(D)$ with divisor D. $M := X\backslash D$ is said to be of finite volume, if there exists an AZD h of $K_X + D$ such that*

$$\int_M h^{-1} \cdot h_D$$

is finite, where $h_D := |\sigma_D|^{-2}$. □

Remark 2.15 *In the above definition h is not an AZD of minimal singularities, when $K_X + D$ is ample.* □

Example 2.16 *Let ω_E be a complete Kähler-Einstein form on M such that $-\mathrm{Ric}_{\omega_E} = \omega_E$ ([Ko]). We set $n := \dim X$. Then $h = (\omega_E^n) \cdot h_D$ is an AZD on $K_X + D$ such that*

$$\int_M h^{-1} \cdot h_D = \int_M \omega_E^n < +\infty.$$

Hence M is of finite volume. □

Theorem 2.17 *Let X be a smooth projective variety and let D be a divisor with simple normal crossings on X. We set $M := X \backslash D$. Suppose that M is of finite volume. Then $h_{can,M} \cdot h_D$ is an AZD of $K_X + D$.* □

The following problem seems to be interesting.

Problem 2.18 *Let X be a smooth projective variety and let D be a divisor with only normal crossings on X such that $K_X + D$ is ample. We set $M := X \backslash D$. Is $\hat{h}_{can,M}^{-1}$ a constant multiple of the Kähler-Einstein volume form on X constructed in [Ko] ?* □

If the above problem is affimative $(M, \hat{h}_{can,M}^{-1})$ is of finite volume.

3 Pseudoeffectivity of relative adjoint bundles

The semipositivity of the relative canonical bundles has been explored by many mathematicians. The weakest notion of semipositity is the pseudoeffectivity.

Definition 3.1 *Let (L, h_L) be a singular hermitian $\mathbb{Q}$-line bundle on a complex manifold X. (L, h_L) is said to be pseudoeffective, if the curvature current $\sqrt{-1}\,\Theta_{h_L}$ of h_L is semipositive. And a $\mathbb{Q}$-line bundle L on a complex manifold X is said to be pseudoeffecive, if there exists a singular hermitian metric h_L on L with semipositive curvature.* □

Now we consider the following problem.

Problem 3.2 *Let $f : X \longrightarrow Y$ be a surjective proper projective morphism between complex manifolds. Let D be a pseudoeffective $\mathbb{Q}$-divisor on X. Let $Y^\circ := \{y \in Y | \, f \text{ is smooth over } y\}$. Suppoose that there exists a point $y_0 \in Y^\circ$ such that $K_{X_{y_0}} + D_{y_0}$ is pseudoeffective. Prove that $K_{X/Y} + D$ is pseudoeffective on X.* □

A partial answer to Problem 3.2 is the following theorem.

Theorem 3.3 *([B-P]) Let $f : X \longrightarrow Y$ be a surjective proper projective morphism between complex manifolds. Let (L, h) be a pseudoeffective singular hermitian line bundle on X. Suppose that there exists a point $y_0 \in Y$ such that $H^0(X_{y_0}, \mathcal{O}_{y_0}(K_{X_{y_0}} + L_{y_0}) \otimes \mathcal{I}(h)) \neq 0$. Then $K_{X/Y} + L$ is pseudoeffective on X.* □

If (X, D) is generically KLT. Then the problem has already been settled.

Theorem 3.4 *([T4]) Let $f : X \longrightarrow Y$ be a surjective proper projective morphism between complex manifolds. Let D be a pseudoeffective $\mathbb{Q}$-divisor on X. Suppose that there exists a point $y_0 \in Y$ such that f is smooth over y_0, (X_{y_0}, D_{y_0}) is KLT and $K_{X_{y_0}} + D_{y_0}$ is pseudoeffective. Then $K_{X/Y} + D$ is pseudoeffective on X. The same statement holds replacing D by a pseudoeffective KLT $\mathbb{Q}$-line bundle (L, h_L) on X.* □

In the case of Kähler morphism, the following conjecture seems to be reasonable.

Conjecture 3.5 *Let $f : X \longrightarrow Y$ be a surjective proper Kähler morphism between complex manifolds. Suppose that there exists a point $y_0 \in Y$ such that f is smooth over y_0 and $K_{X_{y_0}}$ is pseudoeffective. Then $K_{X/Y}$ is pseudoeffective.* □

The above conjecture follows from the following well known conjecture.

Conjecture 3.6 *Let X be a compact Kähler manifold. Suppose that K_X is not pseudoeffective. Then X is uniruled.* □

In the case of smooth projective varieties this conjecture is affirmative by using Mori theory. A $(1,1)$-cohomology calss α on a compact Kähler manifold is said to be nef, if for a fixed Kähler form ω on X, $\alpha + \varepsilon[\omega]$ is represented by another Kähler form for every $\varepsilon > 0$. The following conjecture is also known in the case of smooth projective varieties.

Conjecture 3.7 *Let X be a compact Kähler manifold. Suppose that K_X is not nef. Then X contains a rational curve.* □

4 Adjoint line bundles

Let X be a smooth projective variety and let L be an ample line bundle on X. We call the pair (X, L) a polarized manifold. The basic invariant of the polarized manifold (X, L) is the Hilbert polynomial $P_{(X,L)}(m)$ defined by

$$P_{(X,L)}(m) := \chi(X, \mathcal{O}_X(mL)). \tag{4.0.4}$$

The following vague question is important.

Problem 4.1 *Find any restriction on the Hilbert polynomials.* □

The Hilbert polynomial $P_{(X,L)}(m)$ is of the form:

$$P_{(X,L)}(m) = \frac{L^n}{n!}m^n - \frac{K_X \cdot L^{n-1}}{2(n-1)!}m^{n-1} + \cdots .$$

It is known that if we fix L^n and $K_X \cdot L^{n-1}$, then such a (X, L) forms a bounded family. Hence $P_{(X,L)}(m)$ is rather restrictive. But no so much is known. Following conjecture is due to Y. Kawamata.

Conjecture 4.2 *([Ka1]) Let X be a smooth projective variety and let D be an effective $\mathbb{Q}$-divisor on X such that (X, D) is KLT. Let L be a nef line bundle on X such that $L - (K_X + D)$ is ample. Then $H^0(X, \mathcal{O}_X(L)) \neq 0$ holds.* □

Following is a special case of Conjecture 4.2.

Problem 4.3 *Let X be a smooth projective Fano variety, then $|-K_X| \neq \emptyset$.* □

Problem 4.4 *Let X be a smooth projective variety and let L be an ample line bundle on X. Then the Hilbert polynomial $P_{(X,L)}(m) := \chi(X, \mathcal{O}_X(K_X + mL))$ is nondecreasing for $m \geqq 1, m \in \mathbb{Z}$.* □

If the Problem 4.4 is affirmative, then we see that $h^0(X, \mathcal{O}_X(K_X + mL)) > 0$ for every $m \geqq n+1$.

The following problem is very interesting but it seems to be very hard to solve. And it might be too bold.

Problem 4.5 *Let X be a smooth projective variety and let L be an ample line bundle on X. Then*

$$h^0(X, \mathcal{O}_X(K_X + L)) \geqq h^0(X, \mathcal{O}_X(K_X))$$

holds. □

The following is a special case of Problem 4.4.

Problem 4.6 *Let X be a smooth canonically polarized variety. Then*

$$P_m(X) := h^0(X, \mathcal{O}_X(mK_X))$$

is nondecreasing for $m \geqq 1, m \in \mathbb{Z}$. □

Through the L^2-index theorem, Problem 4.6 is related to the following pointwise problem, if X is covered by a bounded pseudoconvex domain in the case of $m \geqq 2$.

Problem 4.7 *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We set $K_0 = 1$ and*

$$K_{m+1} = K((m+1)K_\Omega, K_m^{-1})$$

for $m \geqq 1$. Prove that $V_m := K_m^{-1} \cdot K_{m+1}$ is monotone increasing. □

The Problem 4.7 is affirmative in the case that Ω is a bounded hermitian symmetric domain.

Theorem 4.8 *Let D be a bounded symmetric domain and let X be a compact quotient of D by a torsion free lattice. Then $P(m) := h^0(X, \mathcal{O}_X(mK_X))$ is monotone increasing for $m \geqq 2$.* □

The following conjecture is due to T. Fujita.

Conjecture 4.9 *(Sectional genus conjecture) Let (X, L) be a polarized manifold of dimension n. Prove that*

$$h^1(X, \mathcal{O}_X) \leqq \frac{(K_X + (n-1)L) \cdot L^{n-1}}{2} + 1$$

holds. □

This conjecture is trivial when L is very ample. In fact the conjecure follows from Lefschetz hyperplane section theorem. One can see that Problem 4.5 is equivalent to the Conjecture 4.9 in the case of $\dim X = 2$.

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